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Generalized Octonion Electrodynamics

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Abstract We have made an attempt to reformulate the generalized field equation of dyons in terms of octonion variables. Octonion forms of generalized potential and current equations are discussed in consistent manner. It has been shown that due to the non associativity of octonion variables it is necessary to impose certain constraints to describe generalized octonion electrodynamics in manifestly covariant and consistent manner.

Keywords Octonion · Dyons · Electrodynamics

1 Introduction

There has been a revival in the formulation of natural laws so that there exists [1] four-division algebras consisting the algebra of real numbers ($\mathbb{R}$), complex numbers ($\mathbb{C}$), quaternions ($\mathbb{H}$) and Octonions ($\mathcal{O}$). All four algebra's are alternative with totally anti symmetric associators. Quaternions [2, 3] were very first example of hyper complex numbers have been widely used [4–15] to the various applications of mathematics and physics. Since octonions [16, 17] share with complex numbers and quaternions, many attractive mathematical properties, one might except that they would be equally as useful as others. Octonion [16, 17] analysis has been widely discussed by Baez [18]. It has also played an important role in the context of various physical problems [19–37] of higher dimensional supersymmetry, super gravity and super strings etc. In recent years, it has also drawn interests of many [38–46] towards the developments of wave equation and octonion form of Maxwell's equations. We [47–53] have also studied octonion electrodynamics, dyonic field equation and octonion gauge analyticity of dyons consistently and obtained the corresponding field equations

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Table 1 Octonion multiplication table

$\cdot$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	e_7	$-e_6$	e_5	$-e_4$
e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	-1	$-e_5$	e_4	e_7	$-e_6$
e_4	$-e_7$	$-e_6$	e_5	-1	$-e_3$	e_2	e_1
e_5	e_6	$-e_7$	$-e_4$	e_3	-1	$-e_1$	e_2
e_6	$-e_5$	e_4	$-e_7$	$-e_2$	e_1	-1	e_3
e_7	e_4	e_5	e_6	$-e_1$	$-e_2$	$-e_3$	-1

(Maxwell's equations) and equation of motion in compact and simpler formulation. Extending our previous results [51–53], in this paper, we have made an attempt to reformulate the generalized field equation of dyons in terms of octonion variables. Octonion forms of generalized potential and current equations are discussed in consistent manner. It has been shown that due to the non associativity of octonion variables it is necessary to impose certain constraints to describe generalized octonion electrodynamics in manifestly covariant and consistent manner. We have obtained the generalized Dirac-Maxwell's (GDM) equations of dyons from octonion wave equation in simple, compact and consistent manner. It has been shown that the octonion variable of dyons reproduces the dynamics of electric (magnetic) charge in the absence of magnetic (electric) charge.

2 Octonion Definition

An octonion x is expressed [47–53] as a set of eight real numbers

$$\begin{aligned} x &= (x_0, x_1, \dots, x_7) = x_0 e_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 + x_5 e_5 + x_6 e_6 + x_7 e_7 \\ &= x_0 e_0 + \sum_{A=1}^{7} x_A e_A \quad (A = 1, 2, \dots, 7), \end{aligned} \tag{1}$$

where e_A $(A = 1, 2, \dots, 7)$ are imaginary octonion units and e_0 is the multiplicative unit element. The octet $(e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7)$ is known as the octonion basis and its elements satisfy the following multiplication rules

$$\begin{aligned} &e_0 = 1, \qquad e_0 e_A = e_A e_0 = e_A, \qquad e_A e_B = -\delta_{AB} e_0 + f_{ABC} e_C \\ &\quad (A, B, C = 1, 2, \dots, 7). \end{aligned} \tag{2}$$

The structure constants f_{ABC} are completely antisymmetric and take the value 1, i.e. $f_{ABC} = +1 = (123), (471), (257), (165), (624), (543), (736)$. Here the octonion algebra $\mathcal{O}$ is described over the algebra of rational numbers having the vector space of dimension 8. Octonion algebra is non associative and multiplication rules for its basis elements given by (2, 3) are then generalized in Table 1:

Hence, we get the following relations among octonion basis elements, i.e.

$$[e_A, e_B] = 2 f_{ABC} e_C; \qquad \{e_A, e_B\} = -\delta_{AB} e_0; \qquad e_A(e_B e_C) \neq (e_A e_B) e_C, \tag{3}$$

where brackets $[\cdot]$ and $\{\cdot\}$ are used respectively for commutation and the anti commutation relations while δ_{AB} is the usual Kronecker delta-Dirac symbol. Octonion conjugate is thus defined as,

$$\bar{x} = x_0 e_0 - x_1 e_1 - x_2 e_2 - x_3 e_3 - x_4 e_4 - x_5 e_5 - x_6 e_6 - x_7 e_7$$
$$= x_0 e_0 - \sum_{A=1}^{7} x_A e_A \quad (A = 1, 2, \dots, 7). \tag{4}$$

An Octonion can be decomposed in terms of its scalar ($Sc(x)$) and vector ($Vec(x)$) parts as

$$Sc(x) = \frac{1}{2}(x + \bar{x}) = x_0; \qquad Vec(x) = \frac{1}{2}(x - \bar{x}) = \sum_{A=1}^{7} x_A e_A. \tag{5}$$

Conjugates of product of two octonions and its own are described as

$$(\overline{xy}) = \overline{y}\overline{x}; \qquad \overline{(\bar{x})} = x \tag{6}$$

while the scalar product of two octonions is defined as

$$\langle x, y \rangle = \sum_{\alpha=0}^{7} x_\alpha y_\alpha = \frac{1}{2}(x\bar{y} + y\bar{x}) = \frac{1}{2}(\bar{x}y + \bar{y}x) \tag{7}$$

which can be written in terms of octonion units as

$$\langle e_A, e_B \rangle = \frac{1}{2}(e_A \overline{e_B} + e_B \overline{e_A}) = \frac{1}{2}(\overline{e_A} e_B + \overline{e_B} e_A) = \delta_{AB}. \tag{8}$$

The norm of the octonion $N(x)$ is defined as

$$N(x) = \bar{x}x = x\bar{x} = \sum_{\alpha=0}^{7} x_\alpha^2 e_0 \tag{9}$$

which is zero if $x = 0$, and is always positive otherwise. It also satisfies the following property of normed algebra

$$N(xy) = N(x)N(y) = N(y)N(x). \tag{10}$$

As such, for a nonzero octonion x, we define its inverse as

$$x^{-1} = \frac{\bar{x}}{N(x)} \tag{11}$$

which shows that

$$x^{-1}x = xx^{-1} = 1.e_0; \qquad (xy)^{-1} = y^{-1}x^{-1}. \tag{12}$$

3 Octonion Wave Equation

A lot of literature [38–46] has already been available on octonion wave equation. Accordingly, let us define the octonion differential operator D as

$$D = \sum_{\mu=0}^{7} e_\mu D_\mu, \tag{13}$$

where D_μ are described as the components of a differential operator in an eight dimensional representation. We describe a function of an octonion variable as

$$\mathcal{F}(X) = \sum_{\mu=0}^{7} e_\mu f_\mu(X) = f_0 + e_1 f_1 + e_2 f_2 + \cdots + e_7 f_7, \tag{14}$$

where f_μ are scalar functions. Since octonions are neither commutative nor associative, one has to be very careful to multiply the octonion either from left or from right in terms of regularity conditions [38–40]. As such, a function $\mathcal{F}(X)$ of an octonion variable $X = \sum_{\mu=0}^{7} e_\mu X_\mu$ is left regular at X if and only if $\mathcal{F}(X)$ satisfies the condition

$$D\mathcal{F}(X) = 0. \tag{15}$$

Similarly, a function $G(X)$ is a right regular if and only if

$$G(X)D = 0, \tag{16}$$

where $G(X) = g_0 + g_1 e_1 + g_2 e_2 + \cdots + g_7 e_7$. Then we get

$$D\mathcal{F} = I = I_0 + I_1 e_1 + I_2 e_2 + I_3 e_3 + I_4 e_4 + I_5 e_5 + I_6 e_6 + I_7 e_7, \tag{17}$$

where

$$\begin{aligned}
I_0 &= \partial_0 f_0 - \partial_1 f_1 - \partial_2 f_2 - \partial_3 f_3 - \partial_4 f_4 - \partial_5 f_5 - \partial_6 f_6 - \partial_7 f_7;\\
I_1 &= \partial_0 f_1 + \partial_1 f_0 + \partial_2 f_3 - \partial_3 f_2 + \partial_6 f_5 - \partial_5 f_6 - \partial_7 f_4 + \partial_4 f_7;\\
I_2 &= \partial_0 f_2 + \partial_2 f_0 + \partial_3 f_1 - \partial_1 f_3 + \partial_4 f_6 - \partial_6 f_4 - \partial_7 f_5 + \partial_5 f_7;\\
I_3 &= \partial_0 f_3 + \partial_3 f_0 + \partial_1 f_2 - \partial_2 f_1 + \partial_6 f_7 - \partial_7 f_6 + \partial_5 f_4 - \partial_4 f_5;\\
I_4 &= \partial_0 f_4 + \partial_4 f_0 + \partial_3 f_5 - \partial_5 f_3 - \partial_2 f_6 + \partial_6 f_2 - \partial_1 f_7 + \partial_7 f_1;\\
I_5 &= \partial_0 f_5 + \partial_5 f_0 + \partial_1 f_6 - \partial_6 f_1 + \partial_7 f_2 - \partial_2 f_7 - \partial_3 f_4 + \partial_4 f_3;\\
I_6 &= \partial_0 f_6 + \partial_6 f_0 - \partial_1 f_5 + \partial_5 f_1 + \partial_2 f_4 - \partial_4 f_2 - \partial_3 f_7 + \partial_7 f_3;\\
I_7 &= \partial_0 f_7 + \partial_7 f_0 + \partial_1 f_4 - \partial_4 f_1 + \partial_2 f_5 - \partial_5 f_2 - \partial_6 f_3 + \partial_3 f_6.
\end{aligned} \tag{18}$$

The regularity condition (15) may now be considered as a homogeneous octonion wave equation for octonion variables without sources. On the other hand, (17) is considered as the inhomogeneous wave equation $D\mathcal{F} = I$.

4 Octonion Formulation for Generalized Fields of Dyons

In order to write the various quantum equations of dyons in octonion formulation, let us start with potential octonion

$$\mathbb{V} = e_0 V_0 + e_1 V_1 + e_2 V_2 + e_3 V_3 + e_4 V_4 + e_5 V_5 + e_6 V_6 + e_7 V_7 \tag{19}$$

and we identify its components as

$$(V_0, V_1, V_2, V_3, V_4, V_5, V_6, V_7) \implies (\varphi, A_x, A_y, A_z, iB_x, iB_y, iB_z, i\phi) \quad (i = \sqrt{-1}), \tag{20}$$

where $(\phi, A_x, A_y, A_z) = (\phi, \overrightarrow{A}) = \{A_\mu\}$ and $(\varphi, B_x, B_y, B_z) = (\varphi, \overrightarrow{B}) = \{B_\mu\}$ are respectively described as the components of electric $\{A_\mu\}$ and magnetic $\{B_\mu\}$ four potentials of dyons (particles carrying simultaneously the electric and magnetic charges). Equation (19) may then be written as

$$\begin{aligned}\mathbb{V} &= e_1(A_x + ie_7 B_x) + e_2(A_y + ie_7 B_y) + e_3(A_z + ie_7 B_z) + (\varphi + ie_7\phi) \\ &= e_1 \mathrm{V}_x + e_2 \mathrm{V}_y + e_3 \mathrm{V}_z + ie_7 \emptyset \end{aligned} \tag{21}$$

where $(\emptyset, \mathrm{V}_x, \mathrm{V}_y, \mathrm{V}_z) = (\emptyset, \overrightarrow{\mathrm{V}}) = \{\mathrm{V}_\mu\}$ are then be described as the components of generalized four potential $\{\mathrm{V}_\mu\}$ associated with generalized charge $(q = e + ig)$ (where e and g are respectively known as electric and magnetic charges) of dyons [6–8]. In order to obtain the generalized field equations of dyons in four dimensional space time, we identify differential operator (13) to be four dimensional and hence we may write (13) as

$$D \longmapsto \boxdot = e_1 \frac{\partial}{\partial x} + e_2 \frac{\partial}{\partial y} + e_3 \frac{\partial}{\partial z} - ie_7 \frac{\partial}{\partial t}, \tag{22}$$

where we have taken other components like $\partial_0, \partial_4, \partial_5, \partial_6$ of (13) vanishing. Octonion conjugate of (22) may then be written as

$$\overline{\boxdot} = -e_1 \frac{\partial}{\partial x} - e_2 \frac{\partial}{\partial y} - e_3 \frac{\partial}{\partial z} + ie_7 \frac{\partial}{\partial t}. \tag{23}$$

Now operating $\overline{\boxdot}$ given by (23) to octonion potential $\mathbb{V}$ of (21), we get

$$\begin{aligned}\overline{\boxdot}\mathbb{V} = &-e_0\left(\overrightarrow{\nabla} \cdot \overrightarrow{A} + \frac{\partial \phi}{\partial t}\right) \\ &+ e_1\left(-\frac{\partial \varphi}{\partial x} + \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} - \frac{\partial B_x}{\partial t}\right) \\ &+ e_2\left(-\frac{\partial \varphi}{\partial y} + \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial zx} - \frac{\partial B_y}{\partial t}\right) \\ &+ e_3\left(-\frac{\partial \varphi}{\partial z} + \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} - \frac{\partial B_z}{\partial t}\right) \\ &- ie_4\left(-\frac{\partial \phi}{\partial x} - \frac{\partial B_z}{\partial y} + \frac{\partial B_y}{\partial z} - \frac{\partial A_x}{\partial t}\right) \\ &- ie_5\left(-\frac{\partial \phi}{\partial y} - \frac{\partial B_x}{\partial z} + \frac{\partial B_z}{\partial x} - \frac{\partial A_y}{\partial t}\right) \\ &- ie_6\left(-\frac{\partial \phi}{\partial z} - \frac{\partial B_y}{\partial x} + \frac{\partial B_x}{\partial y} - \frac{\partial A_z}{\partial t}\right) \\ &+ ie_7\left(\overrightarrow{\nabla} \cdot \overrightarrow{B} + \frac{\partial \varphi}{\partial t}\right). \end{aligned} \tag{24}$$

We are using S.I. system of natural units $(c = \hbar = 1)$. On applying the Lorentz gauge conditions, respectively for the dynamics of electric and magnetic charges of dyons as

$$\begin{aligned}\overrightarrow{\nabla}\cdot\overrightarrow{A}+\frac{\partial\phi}{\partial t}&=0;\\ \overrightarrow{\nabla}\cdot\overrightarrow{B}+\frac{\partial\varphi}{\partial t}&=0,\end{aligned}\tag{25}$$

we find the following octonion form of (24), i.e.

$$\overline{\boxdot}\mathbb{V}=\mathbb{F},\tag{26}$$

where $\mathbb{F}$ is again an octonion reproduces the generalized electromagnetic fields of dyons. It is thus described by

$$\mathbb{F}=e_0F_0+e_1F_1+e_2F_2+e_3F_3+e_4F_4+e_5F_5+e_6F_6+e_7F_7,\tag{27}$$

where

$$\begin{aligned}F_0&=-\left(\overrightarrow{\nabla}\cdot\overrightarrow{A}+\frac{\partial\phi}{\partial t}\right)=0;\\ F_1&=\left(-\frac{\partial\varphi}{\partial x}+\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}-\frac{\partial B_x}{\partial t}\right)=H_x;\\ F_2&=\left(-\frac{\partial\varphi}{\partial y}+\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}-\frac{\partial B_y}{\partial t}\right)=H_y;\\ F_3&=\left(-\frac{\partial\varphi}{\partial z}+\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}-\frac{\partial B_z}{\partial t}\right)=H_z;\\ F_4&=-i\left(-\frac{\partial\phi}{\partial x}-\frac{\partial B_z}{\partial y}+\frac{\partial B_y}{\partial z}-\frac{\partial A_x}{\partial t}\right)=-iE_x;\\ F_5&=-i\left(-\frac{\partial\phi}{\partial y}-\frac{\partial B_x}{\partial z}+\frac{\partial B_z}{\partial x}-\frac{\partial A_y}{\partial t}\right)=-iE_y;\\ F_6&=-i\left(-\frac{\partial\phi}{\partial z}-\frac{\partial B_y}{\partial x}+\frac{\partial B_x}{\partial y}-\frac{\partial A_z}{\partial t}\right)=-iE_z;\\ F_7&=i\left(\overrightarrow{\nabla}\cdot\overrightarrow{B}+\frac{\partial\varphi}{\partial t}\right)=0.\end{aligned}\tag{28}$$

Let us define [6–8, 47–50] the generalized electric ($\overrightarrow{E}$) and magnetic ($\overrightarrow{H}$) fields of dyons in terms of components of electric and magnetic four potentials as

$$\begin{aligned}\overrightarrow{E}&=-\frac{\partial\overrightarrow{A}}{\partial t}-\overrightarrow{\nabla}\phi-\overrightarrow{\nabla}\times\overrightarrow{B};\\ \overrightarrow{H}&=-\frac{\partial\overrightarrow{B}}{\partial t}-\overrightarrow{\nabla}\varphi+\overrightarrow{\nabla}\times\overrightarrow{A}.\end{aligned}\tag{29}$$

Thus (27) reduces to

$$\begin{aligned}\mathbb{F}&=e_1(H_x+ie_7E_x)+e_2(H_y+ie_7E_y)+e_3(H_z+ie_7E_z)\\ &=e_1\Psi_x+e_2\Psi_y+e_3\Psi_z,\end{aligned}\tag{30}$$

where $\overrightarrow{\Psi} = \overrightarrow{H} + ie_7\overrightarrow{E}$ is the generalized vector field [6–8, 47–50] of dyons. Now applying the differential operator (22) to (30), we get

$$\begin{aligned}\boxdot\mathbb{F} = &-e_0\left(\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z}\right)\\ &+ e_1\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - \frac{\partial E_x}{\partial t}\right)\\ &+ e_2\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} - \frac{\partial E_y}{\partial t}\right)\\ &+ e_3\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - \frac{\partial E_z}{\partial t}\right)\\ &+ ie_4\left(\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} - \frac{\partial H_x}{\partial t}\right)\\ &+ ie_5\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} - \frac{\partial H_y}{\partial t}\right)\\ &+ ie_6\left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} - \frac{\partial H_z}{\partial t}\right)\\ &+ ie_7\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)\end{aligned} \tag{31}$$

or

$$\begin{aligned}\boxdot\mathbb{F} = &-e_0(\overrightarrow{\nabla}.\overrightarrow{H})\\ &+ e_1\left[(\overrightarrow{\nabla}\times\overrightarrow{H})_x - \frac{\partial E_x}{\partial t}\right]\\ &+ e_2\left[(\overrightarrow{\nabla}\times\overrightarrow{H})_y - \frac{\partial E_y}{\partial t}\right]\\ &+ e_3\left[(\overrightarrow{\nabla}\times\overrightarrow{H})_z - \frac{\partial E_z}{\partial t}\right]\\ &+ ie_4\left[(\overrightarrow{\nabla}\times\overrightarrow{E})_x - \frac{\partial H_x}{\partial t}\right]\\ &+ ie_5\left[(\overrightarrow{\nabla}\times\overrightarrow{E})_y - \frac{\partial H_y}{\partial t}\right]\\ &+ ie_6\left[(\overrightarrow{\nabla}\times\overrightarrow{E})_z - \frac{\partial H_z}{\partial t}\right]\\ &+ ie_7(\overrightarrow{\nabla}.\overrightarrow{E}).\end{aligned} \tag{32}$$

These (31, 32) may then be written in following compact notation in terms of an octonion, i.e.

$$\boxdot\mathbb{F} = \mathbb{J}, \tag{33}$$

where $\mathbb{J}$ is again an octonion and is re-described as the Octonion form of generalized current given by

$$\begin{aligned}\mathbb{J} &= -e_0\varrho + e_1\mathrm{j}_x + e_2\mathrm{j}_y + e_3\mathrm{j}_z - \mathrm{i}e_4\mathrm{k}_x - \mathrm{i}e_5\mathrm{k}_y - \mathrm{i}e_6\mathrm{k}_z + \mathrm{i}e_7\rho \\ &= (e_1\mathrm{j}_x + e_2\mathrm{j}_y + e_3\mathrm{j}_z - e_0\varrho) + i(e_1\mathrm{k}_x + e_2\mathrm{k}_y + e_3\mathrm{k}_z - \rho)e_7 \\ &= e_1(\mathrm{j}_x + ie_7k_x) + e_2(\mathrm{j}_y + \mathrm{i}e_7\mathrm{k}_y) + e_3(\mathrm{j}_z + \mathrm{i}e_7\mathrm{k}_z) - (\rho + \mathrm{i}e_7\varrho) \\ &= e_1\mathrm{J}_x + e_2\mathrm{J}_y + e_3\mathrm{J}_z + \mathrm{i}e_7\mathrm{J}_0,\end{aligned} \tag{34}$$

where $(\rho, \overrightarrow{j}) = \{\mathrm{j}_\mu\}$, $(\varrho, \overrightarrow{j}) = \{\mathrm{k}_\mu\}$ and $(\mathrm{J}_0, \overrightarrow{J}) = \{\mathrm{J}_\mu\}$ are respectively the four currents associated with electric charge, magnetic monopole and generalized fields of dyons. Equations (31–34) thus lead to following differential equations

$$\begin{aligned}(\overrightarrow{\nabla} \cdot \overrightarrow{H}) &= \varrho; \\ (\overrightarrow{\nabla} \times \overrightarrow{H})_x &= \frac{\partial E_x}{\partial t} + \mathrm{j}_x; \\ (\overrightarrow{\nabla} \times \overrightarrow{H})_y &= \frac{\partial E_y}{\partial t} + \mathrm{j}_y; \\ (\overrightarrow{\nabla} \times \overrightarrow{H})_z &= \frac{\partial E_z}{\partial t} + \mathrm{j}_z; \\ (\overrightarrow{\nabla} \times \overrightarrow{E})_x &= -\frac{\partial H_x}{\partial t} - \mathrm{k}_x; \\ (\overrightarrow{\nabla} \times \overrightarrow{E})_y &= -\frac{\partial H_y}{\partial t} - \mathrm{k}_y; \\ (\overrightarrow{\nabla} \times \overrightarrow{E})_z &= -\frac{\partial H_z}{\partial t} - \mathrm{k}_z; \\ (\overrightarrow{\nabla} \cdot \overrightarrow{E}) &= \rho.\end{aligned} \tag{35}$$

Equation (35) may then be written as

$$\begin{aligned}(\overrightarrow{\nabla} \cdot \overrightarrow{E}) &= \rho; \\ (\overrightarrow{\nabla} \times \overrightarrow{E}) &= -\frac{\partial \overrightarrow{H}}{\partial t} - \overrightarrow{k}; \\ (\overrightarrow{\nabla} \times \overrightarrow{H}) &= \frac{\partial \overrightarrow{E}}{\partial t} + \overrightarrow{j}; \\ (\overrightarrow{\nabla} \cdot \overrightarrow{H}) &= \varrho\end{aligned} \tag{36}$$

which are the generalized Dirac-Maxwell's equations of generalized fields of dyons [47–53].

$$\begin{aligned}\boxdot\bar{\boxdot}\mathbb{V} &= \mathbb{F} \\ &= e_0\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} + \frac{\partial^2 B_x}{\partial x\partial t} + \frac{\partial^2 B_y}{\partial y\partial t} + \frac{\partial^2 B_z}{\partial z\partial t}\right) \\ &\quad - e_1\left(\frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2} - \frac{\partial^2 A_x}{\partial t^2} - \frac{\partial^2 A_z}{\partial x\partial z} - \frac{\partial^2 A_y}{\partial y\partial x} - \frac{\partial^2 \phi}{\partial x\partial t}\right)\end{aligned}$$

$$- e_2\left(\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial z^2} - \frac{\partial^2 A_y}{\partial t^2} - \frac{\partial^2 A_x}{\partial y \partial x} - \frac{\partial^2 A_z}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial y \partial t}\right)$$

$$- e_3\left(\frac{\partial^2 A_z}{\partial z^2} + \frac{\partial^2 A_z}{\partial y^2} - \frac{\partial^2 A_z}{\partial t^2} - \frac{\partial^2 A_y}{\partial z \partial y} - \frac{\partial^2 A_x}{\partial z \partial x} - \frac{\partial^2 \phi}{\partial z \partial t}\right)$$

$$+ ie_4\left(-\frac{\partial^2 B_x}{\partial y^2} - \frac{\partial^2 B_x}{\partial z^2} + \frac{\partial^2 B_x}{\partial t^2} + \frac{\partial^2 B_y}{\partial x \partial y} + \frac{\partial^2 B_z}{\partial z \partial x} + \frac{\partial^2 \varphi}{\partial x \partial t}\right)$$

$$+ ie_5\left(-\frac{\partial^2 B_y}{\partial x^2} - \frac{\partial^2 B_y}{\partial z^2} + \frac{\partial^2 B_y}{\partial t^2} + \frac{\partial^2 B_z}{\partial z \partial y} + \frac{\partial^2 B_x}{\partial y \partial x} + \frac{\partial^2 \varphi}{\partial y \partial t}\right)$$

$$+ ie_6\left(-\frac{\partial^2 B_z}{\partial x^2} - \frac{\partial^2 B_z}{\partial y^2} + \frac{\partial^2 B_z}{\partial t^2} + \frac{\partial^2 B_x}{\partial x \partial z} + \frac{\partial^2 B_y}{\partial z \partial y} + \frac{\partial^2 \varphi}{\partial z \partial t}\right)$$

$$- ie_7\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 A_x}{\partial x \partial t} + \frac{\partial^2 A_y}{\partial t \partial y} + \frac{\partial^2 A_z}{\partial z \partial t}\right). \tag{37}$$

Equation (37) then reduces to

$$\boxdot\, \bar{\boxdot}\, \mathbb{V} = \bar{\boxdot}\, \boxdot\, \mathbb{V} = \mathbb{J}, \tag{38}$$

where $\mathbb{J}$ is described as the octonion form of generalized current associated with dyon and is already given by (33). Equation (37) may also be written as

$$\boxdot\, \bar{\boxdot}\, \mathbb{V} = \bar{\boxdot}\, \boxdot\, \mathbb{V} = e_0\left[\nabla^2 \varphi + \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{B})\right]$$

$$- e_1\left[\Box A_x - \frac{\partial}{\partial x}\left(\vec{\nabla} \cdot \vec{A} + \frac{\partial \phi}{\partial t}\right)\right]$$

$$- e_2\left[\Box A_y - \frac{\partial}{\partial y}\left(\vec{\nabla} \cdot \vec{A} + \frac{\partial \phi}{\partial t}\right)\right]$$

$$- e_3\left[\Box A_z - \frac{\partial}{\partial z}\left(\vec{\nabla} \cdot \vec{A} + \frac{\partial \phi}{\partial t}\right)\right]$$

$$- ie_4\left[\Box B_x - \frac{\partial}{\partial x}\left(\vec{\nabla} \cdot \vec{B} + \frac{\partial \varphi}{\partial t}\right)\right]$$

$$- ie_5\left[\Box B_y - \frac{\partial}{\partial y}\left(\vec{\nabla} \cdot \vec{B} + \frac{\partial \varphi}{\partial t}\right)\right]$$

$$- ie_6\left[\Box B_z - \frac{\partial}{\partial z}\left(\vec{\nabla} \cdot \vec{B} + \frac{\partial \varphi}{\partial t}\right)\right]$$

$$- ie_7\left[\nabla^2 \phi + \frac{\partial}{\partial t}\left(\vec{\nabla} \cdot \vec{A}\right)\right], \tag{39}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

and

$$\Box = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2} = \nabla^2 - \frac{\partial^2}{\partial t^2}.$$

Using the Lorentz gauge conditions (25) and the definition of octonion valued generalized current of dyon given by (34), we get

$$\begin{aligned} \Box\phi &= \rho; \qquad \Box\varphi = \varrho; \\ \Box \mathrm{A}_\mu &= \mathrm{j}_\mu; \qquad \Box \mathrm{B}_\mu = \mathrm{k}_\mu. \end{aligned} \tag{40}$$

As such, we have obtained consistently the generalized Dirac Maxwell's (GDM) equations from octonion wave equations on considering the non associativity of octonion variables. The advantages of present formalism are discussed in terms of compact and simpler notations of octonion valued potential, field and currents of dyons despite of non associativity of octonions. The present octonion reformulation of generalized fields of dyons represents well the invariance of field equations under Lorentz and duality transformations. It also reproduces the dynamics of electric (magnetic) charge yielding to the usual form of Maxwell's equations in the absence of magnetic (electric charge) in compact, simpler and consistent way. Octonion element e_7 has been considered to be invariant and the theory resembles with the bi-quaternion theory of generalized fields of dyons [12]. In the forthcoming paper theory of split octonion will be discussed.

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